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1993 J. Phys. A: Math. Gen. 26 L225

(http://iopscience.iop.org/0305-4470/26/5/009)

J. Phys. A: Math. Gen. 26 (1993) L225–L227. Printed in the UK

LETTER TO THE EDITOR

Polynomial translation modules and Casimir invariants

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Received 4 January 1993

Abstract. We give a positive answer to a question about the existence of any direct link between two apparently unrelated facts for a specific family of solvable Lie algebras: the structure of their modules of functions on one side, and the algebraic form of their Casimir invariants on the other.

In a recent paper [1] (see also [2] and references therein), González López *et al* have obtained a complete classification of finite-dimensional Lie algebras of first-order differential operators in two complex variables, i.e. operators of the form $D = f_1(x, y)\partial_x + f_2(x, y)\partial_y + g(x, y)$.

Let us recall the three basic steps involved (the terminology used closely follows that of reference [1]):

(i) First of all, the classification of all finite-dimensional Lie algebras of vector fields of the form $v = f_1(x, y)\partial_x + f_2(x, y)\partial_y$ is required. This was already achieved by S Lie in his classical work [3]. He found 24 classes $\mathfrak{h}_i$ $(i = 1, \ldots, 24)$ of such Lie algebras, some of which depend on parameters.

Here we will be concerned with two of them, namely,

$$\mathfrak{h}_5(\alpha) = \{\partial_x, \partial_x, x\partial_x + \alpha y\partial_y\} \qquad \alpha \neq 0$$

$$\mathfrak{h}_{20}(\alpha, r) = \{\partial_x, \partial_y, x\partial_x + \alpha y\partial_y, x\partial_y, \ldots, x^r\partial y\} \qquad r \geqslant 1.$$

(ii) Since the Lie bracket of two first order differential operators $D = v + g$, $D' = v' + g'$ involves not only the commutator of the vector fields v, v' but, in addition, crossed terms in which the functions g, g' are acted upon by these vector fields, one needs to classify the finite-dimensional $\mathfrak{h}_i$-modules of C^∞ functions for each of the 24 Lie algebras above.

(iii) Finally, one has to determine, for each $\mathfrak{h}_i$-module M, the first cohomology group of $\mathfrak{h}_i$ with coefficients in $C^\infty(\mathbb{C}^2)/M$.

The purpose of this letter is to provide an affirmative answer to an intriguing question raised in [1]. It is related to the solution of step (ii) for the family of solvable Lie algebras $\mathfrak{h}_5(\alpha)$. Let us begin by recalling some relevant definitions.

Definition 1. The finite-dimensional $\{\partial_x, \partial_y\}$-modules are called translation modules. If such a module is spanned by polynomials (resp. monomials) then it is called a polynomial (resp. monomial) translation module.

Definition 2. A polynomial is called α-homogeneous if the vector field $x\partial_x + \alpha y\partial_y$ takes it into a multiple of itself.

0305-4470/93/050225+03$07.50

Definition 3. A polynomial module is called α-homogeneous if it admits a basis of α-homogeneous polynomials.

The following result makes explicit the structure of the $\mathfrak{H}_5(\alpha)$-modules:

Proposition 1 [1]. Any $\mathfrak{H}_5(\alpha)$-module ($\alpha \neq 0$) is an α-homogeneous polynomial translation module. Moreover, if α is not a positive rational number, then the module is a monomial translation module.

Example [1]: The polynomials

$$x^4+x^2y, 4x^3+2xy, x^2, x, y, 1$$

span a non-monomial $\mathfrak{H}_5(\alpha)$-module for $\alpha=2$.

The proposition above suggests that the dependence of the algebraic structure of the $\mathfrak{H}_5(\alpha)$-modules on α being rational or not, could be related to the fact that these Lie algebras admit polynomial or rational Casimir invariants if and only if α is rational. This is the explicit question formulated in [1]. In order to provide a satisfactory answer we make free use of a few definitions and results from [4].

In the basis $A_1=\partial_x$, $A_2=\partial_y$, $A_3=x\partial_x+\alpha y\partial_y$, the structure of the Lie algebra $\mathfrak{H}_5(\alpha)$ is given by $[A_1, A_2]=0$, $[A_1, A_3]=A_1$, $[A_2, A_3]=\alpha A_2$. Let us denote by S the symmetric algebra of $\mathfrak{H}_5(\alpha)$ and by $D(S)$ the associated quotient field. By definition, rational formal Casimir invariants are the elements of $D(S)'=\{h\in D(S)\colon \hat{A}_j(h)=0, \forall j\}$, where $\hat{A}_1=a_1\partial_{a_3}$, $\hat{A}_2=\alpha a_2\partial_{a_3}$, $\hat{A}_3=-a_1\partial_{a_1}-\alpha a_2\partial_{a_2}$.

Under the canonical isomorphism, $D(S)'$ turns out to be isomorphic to the set $D(U)'$ of rational Casimir invariants, where $D(U)$ denotes the quotient field of the universal enveloping algebra U.

Consider now $S_\lambda^{I/2}=\{h\in S\colon \hat{A}_1(h)=\hat{A}_2(h)=0,\ \hat{A}_3(h)=\lambda h\}$, the set of semi-invariants of weight λ in the symmetric algebra. It is worth noticing that $h\in S_\lambda^{I/2}$ if and only if $\partial_{a_3}h=0$ and $h(a_1, a_2)$ is α-homogeneous with a degree of homogeneity equal to $-\lambda$.

Moreover, we know from [4] that $f\in D(S)^I \Leftrightarrow f=h_1/h_2$ where $h_1, h_2\in S_\lambda^{I/2}$, for some λ.

As we will show in a moment, these two facts are responsible for the required link between the existence of rational Casimir invariants and the structure of the $\mathfrak{H}_5(\alpha)$-modules.

Proposition 2. For the Lie algebra $\mathfrak{H}_5(\alpha)$ ($\alpha\neq 0$) the following statements are equivalent:

(a) α is positive rational;

(b) there are strictly rational (i.e. rational and not polynomial) Casimir invariants;

(c) there are non-monomial $\mathfrak{H}_5(\alpha)$-modules.

Proof: $[(a)\Leftrightarrow(b)]$. This immediately follows from the method explained in [4]. As a matter of fact, $\mathfrak{H}_5(\alpha)$ admits a unique (algebraically independent) formal Casimir invariant $\hat{C}=a_1^\alpha a_2^{-1}$.

$[(b)\Rightarrow(c)]$. Let us assume that there exists a strictly rational formal Casimir invariant $\hat{C}=a_1^m a_2^{-n}$, i.e. $\alpha=m/n$, where m, n are two different positive integers. Then it is very easy to verify that, under the action of ∂_x, ∂_y, the function x^m+y^n generate the polynomials x^m+y^n, $x^{m-1}, \ldots, x$, $y^{n-1}, \ldots, y, 1$ which constitute the basis of a non-monomial $\mathfrak{H}_5(\alpha)$-module.

Remark: The same is true for $q(x,y)(x^m+y^n)$, where q stands for an arbitrary m/n-homogeneous polynomial. In fact, the example above, quoted from [1], corresponds to the particular choice $q(x,y)=x^2$ in the case $m=2$, $n=1$.

[$(c)\Rightarrow(b)$]. Suppose that there is a non-monomial $\mathfrak{H}_5(\alpha)$-module M, and let $h(x, y)$ be a polynomial in its basis. All the monomials in $h(x, y)$ are α-homogeneous to the same degree. Therefore, they all are semi-invariant polynomials with the same weight. Hence, the quotient of any two of them must be a power of the unique (algebraically independent) Casimir invariant. The proof will be complete if we show that in the case $\hat{C}=a_1^m a_2^n$, i.e. $\alpha=-m/n$ is negative rational, $h(x, y)$ can be replaced as the basis of M by its monomials.

Let us define $C=x^m y^n$ and $D_C\equiv D_x^m D_y^n$. Consider a polynomial $h(x, y)=x^{i_0}y^{j_0}C^s[a_0+a_{N_1}C^{N_1}+\cdots+a_{N_k}C^{N_k}]$ as the basis of M with $0<N_1<\cdots<N_k$, $a_{N_i}\neq 0$, $\forall i$, and $D_C(x^{i_0}y^{j_0})=0$. Clearly, $(D_C^{s+N_k}h)(x, y)$ is proportional to $x^{i_0}y^{j_0}$. Thus, this monomial can eventually be incorporated to the basis and simultaneously eliminated from h. Now, up to a multiple of $x^{i_0}y^{j_0}$, $(D_C^{s+N_k-1}h)(x, y)$ is proportional to $x^{i_0}y^{j_0}C$. Consequently, any term of the last form can also be eliminated from h. By continuing this argument we get the desired result.

Finally, a brief comment on $\mathfrak{H}_{20}(\alpha, r)$ is in order. Since it contains $\mathfrak{H}_5(\alpha)$ as a Lie subalgebra, any $\mathfrak{H}_{20}(\alpha, r)$-module is a $\mathfrak{H}_5(\alpha)$-module. In fact, the above construction, starting from x^m+y^n leads, under the action of $\partial_x, \partial_y, x\partial_y, \ldots, x^r\partial_y$, to a basis of a polynomial $\mathfrak{H}_{20}(\alpha, r)$-module. Moreover, it is non-monomial whenever $r<\alpha$.

For instance, x^3+y does the job in the case $\alpha=3$. In fact, the example exhibited in [1] is defined by the basis generated in this sense by $x(x^3+y)$.

It is to be emphasized that, contrary to the case of $\mathfrak{H}_5(\alpha)$, the Casimir invariants of $\mathfrak{H}_{20}(\alpha, r)$ (easy to find by the method in [4]) seems to have no direct influence on the structure of its modules. As far as the module structure is concerned, it can be said that $\mathfrak{H}_{20}(\alpha, r)$ is 'subordinated' to $\mathfrak{H}_5(\alpha)$. Hierarchies of this type might play a significant role in classification problems in more than two variables.

We wish to thank Dr A González López for some valuable suggestions and acknowledge partial financial support by the CICYT. One of us (CMO) thanks the Universidad Complutense for a Grant.

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